

Solution of Non-linear Equations using Bisection Method by New Technical Method

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Abstract:

Numerical approximation of the root-finding problem its important tool for process involves finding a root, or solution of nonlinear equation of the form $f(x) = 0$, for a given function f . A root of this equation is also called a zero of the function f . When we implementing the method on a computer we need to consider the effects of round-off error. For example the computation of the midpoint of the interval $[a_n, b_n]$ should be found from the equation . The Bisection method is used to determine to any specified accuracy that your computer will permit a solution to $f(x) = 0$ on an interval $[a, b]$, provided that f is continuous on the interval and that $f(a)$ and $f(b)$ are of opposite sign. Although the method will work for the case when more than one root is contained in the interval $[a, b]$, we assume for simplicity of our discussion that the root in this interval is unique. the method stops if one of the midpoints happens to coincide with the root. It also stops when the length of the search interval is less than some prescribed tolerance. The having method is characterized by the fact that it always includes convergence of the individual islands. It is also characterized by the case of calculating errors, but one of its disadvantages is that it is slow to converge to reach the solution. To compare with the a new technical method of the solution . We followed applied numerical method using a new technical method in computer and we found that the new technical method of solution is much faster and more accurate.

Keywords : Solution , Non-linear Equations , Bisection Method , New Technical Method.

1. Introduction:

Non-linear equations arise in most disciplines including computer science, natural sciences (civil engineering, electrical engineering, mechanical engineering), biological engineering and social sciences

(psychology, economics), etc. Problems such as minimization, target shooting, orbital motion, plenary motion, etc., often lead to finding roots of non-linear functional equations. There are classical root-finding algorithms: bisection, false position, Newton–Raphson, modified Newton–Raphson, secant and modified secant method, for finding roots of a non-linear equation $f(x) = 0$ [1]. Many bracketing algorithms have been described in literature for finding a single root of a nonlinear equation $f(x) = 0$. The most basic bracketing method is a dichotomy method also known as a bisection method with a rather slow convergence [2]. Bisection method and main results The goal of the bisection method which is studied in this paper is to locate and to approximate the zeros of an analytic function f in a specified bounded domain [3].

2. Solving Nonlinear Equations:

Given a function $f : R \rightarrow R$, we seek a point $x_m \in R$ such that $f(x_m) = 0$. This x_m is called a root of the equation $f(x) = 0$, or simply a zero of f . At first, we only require that f be continuous on an interval $[a, b]$ of the real line, $f \in C[a, b]$, and that this interval contains the root of interest. The function f could have many different roots; we only look for one. In practice, f could be quite complicated (e.g., evaluation of a parameter dependent integral or differential equation) that is expensive to evaluate (e.g., requiring minutes, hours, . . .), so we seek algorithms that produce a solution that is accurate to high precision while keeping evaluations of f to a minimum [4].

3. Definitions and Sources of Error:

We now give a rough classification of the major ways in which error is introduced into the solution of a problem, including some that fall outside the usual scope of mathematics. We begin with a few simple definitions about error. In solving a problem, we seek an exact or true solution, which we denote by

X_T -Approximations are usually made in solving the problem, resulting in an approximate solution X_A . We define the error in X_A by

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$$\text{Error}(X_A) = \text{Error } X_A = X_T - X_A$$

For many purposes, we prefer to study the percentage or relative error in X_A .

$$\text{Rel}(X_A) \equiv \text{relative error in } X_A = \frac{X_T - X_A}{X_T}$$

Provided $X_T \neq 0$. This has already been referred to in $X_A \frac{x-f(x)}{x} \equiv \varepsilon$ in measuring the error in $f(x)$ [5].

4. Bracketing Methods:

In the rest of the text, let f be a real and continuous function on an interval $[a, b]$, $f(a)$ and $f(b)$ have different parity i.e. $f(a)f(b) < 0$. Therefore, there is at least one real root r in the interval $[a, b]$ of the equation $f(x) = 0$ [6].

5. Bisection Method:

This method is based if a function $f(x)$ is continuous between a and b and $f(a)$ and $f(b)$ are of opposite signs then there exists at least one root between a and b . For definition let $f(a)$ be negative and $f(b)$ be positive [7].

We first consider the Bisection (Binary search) Method which is based on the Intermediate Value Theorem (IVT). IVT Illustration Suppose a continuous function, defined on $[a, b]$ is given with $f(a)$ and $f(b)$ of opposite sign. By the IVT, there exists a point $p \in (a, b)$ for which $f(p) = 0$. In what follows, it will be assumed that the root in this interval is unique. Suppose f is a continuous function defined on the interval $[a, b]$, with $f(a)$ and $f(b)$ of opposite sign. The Intermediate Value Theorem implies that a number p exists in (a, b) with $f(p) = 0$. Although the procedure will work when there is more than one root in the interval (a, b) , we assume for simplicity that the root in this interval is unique. The method calls for a repeated halving (or bisecting) of subintervals of $[a, b]$ and, at each step, locating the half containing p [8].

Example (5.1): Find the root of the equation $x^3 - 2x - 5 = 0$ using Bisection method hence the root lies between 2 and 3

Solution:

$$f(2) = -1 \text{ and } f(3) = 16$$

hence the root lies between 2 and 3

$$x_1 = \frac{2 + 3}{2} = 2.5$$

The root lies between 2 and 2.5

Since $f(x_1) = f(2.5) = 5.6250$,

$$x_2 = \frac{2 + 2.5}{2} = 2.25$$

The root lies between 2 and 2.25

Now, $f(x_2) = f(2.25) = 1.890625$,

$$x_3 = \frac{2 + 2.25}{2} = 2.125$$

The root lies between 2 and 2.125

Since $f(x_3) = f(2.125) = 0.3457$,

$$x_4 = \frac{2 + 2.125}{2} = 2.0625$$

Proceeding in this way, we obtain the successive approximations:

$$x_5 = 2.09375 \quad , \quad x_6 = 2.10938$$

$$x_7 = 2.10156 \quad x_8 = 2.09766$$

$$x_9 = 2.09570 \quad x_{10} = 2.09473$$

$$x_{11} = 2.09424 \quad \dots \dots \dots$$

We find

$$x_{11} - x_{10} = -0.0005$$

and

$$\left| \frac{x_{11} - x_{10}}{x_{11}} \right| \times 100 = \frac{0.0005}{2.09424} \times 100 = 0.02\%$$

Hence a root, correct to three decimal places is 2.094[7].

Example (5.2) Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$, and use the Bisection method to determine an approximation to the root that is accurate to at least within 10^{-4} .

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Solution:

Because $f(1) = -5$ and $f(2) = 14$ ensures that this continuous function has a root in $[1, 2]$.

For the first iteration of the Bisection method we use the fact that at the midpoint of $[1, 2]$ we have $f(1.5) = 2.375 > 0$. This indicates that we should select the interval $[1, 1.5]$ for our second iteration. Then we find that $f(1.25) = -1.796875$ so our new interval becomes $[1.25, 1.5]$, whose midpoint is 1.375. Continuing in this manner gives the values in Table (5.1). After 13 iterations, $p_{13} = 1.365112305$ approximates the root with an error

$$|p - p_{13}| < |b_{14} - a_{14}| = |1.365234375 - 1.365112305| = 0.000122070.$$

Since $|a_{14}| < |p|$, we have

$$\frac{|p - p_{13}|}{|p|} < \frac{|b_{14} - a_{14}|}{|a_{14}|} \leq 9.0 \times 10^{-5}$$

Table N0. (5.1):

n	a_n	b_n	p_n	$f(p_n)$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	-1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	-0.84839
5	1.3125	1.375	1.34375	-0.35098
6	1.34375	1.375	1.359375	-0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	-0.03215
9	1.36328125	1.3671875	1.365234375	0.000072
10	1.36328125	1.365234375	1.364257813	-0.01605
11	1.364257813	1.365234375	1.364746094	-0.00799
12	1.364746094	1.365234375	1.364990235	-0.00396
13	1.364990235	1.365234375	1.365112305	-0.00194

So the approximation is correct to at least within 10^{-4} . The correct value of p to nine decimal places is $p = 1.365230013$. Note that p_9 is closer to p than is the final approximation p_{13} . You might suspect

this is true because $|f(p_9)| < |f(p_{13})|$, but we cannot be sure of this unless the true answer is known.

The Bisection method, though conceptually clear, has significant drawbacks. It is relatively slow to converge (that is, N may become quite large before $|p - p_N|$ is sufficiently small), and a good intermediate approximation might be inadvertently discarded. However, the method has the important property that it always converges to a solution, and for that reason it is often used as a starter for the more efficient methods .[8].

Example(5.3): Use the bisection method to approximate the value of $\sqrt{5}$ within 0.00001 accuracy.

Solution:

Finding the square root of 5 is equivalent to finding the roots of $x^2 - 5 = 0$ We expect the positive root to be in the $2 < x < 3$ interval so we assign $x_1 = 2$ $x_2 = 3$. The average of these values is $x_m = 2.5$ we can use it with any polynomial equation. [9]

Table No. (5.2):

n	x_1	x_2	x_m	$f(x_1)$	$f(x_{21})$	$f(x_m)$
1	2.00000	3.00000	2.50000	-1.00000	1.25000	-1.25000
2	2.00000	2.50000	1.25000	-1.00000	0.06250	-0.06250
3	2.00000	2.25000	2.12500	-1.00000	-0.48438	0.48438
4	2.12500	2.25000	2.18750	-0.48438	-0.21484	0.10406
5	2.18750	2.25000	2.21875	-0.21484	-0.07715	0.01657
6	2.21875	2.25000	2.23438	-0.07715	-0.00757	0.00058
7	2.23438	2.25000	2.24219	-0.00757	0.02740	-0.00021
8	2.23438	2.24219	2.23828	-0.00757	0.00990	-0.00007
9	2.23438	2.23828	2.23633	-0.00757	0.00116	-0.00001
10	2.23438	2.23633	2.23535	-0.00757	-0.00320	0.00002
11	2.23535	2.23633	2.23584	-0.00320	-0.00102	0.00000
12	2.23584	2.23633	2.23608	-0.00102	0.00007	0.00000
13	2.23584	2.23608	2.23596	-0.00102	-0.00047	0.00000
14	2.23596	2.23608	2.23602	-0.00047	-0.00020	0.00000
15	2.23602	2.23608	2.23605	-0.00020	-0.00006	0.00000

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16	2.23605	2.23608	2.23607	-0.00006	0.00000	0.00000
17	2.23605	2.23607	2.23606	-0.00006	-0.00003	0.00000
18	2.23606	2.23607	2.23606	-0.00003	-0.00001	0.00000

6. Computational Steps:

To begin, set $a_1 = a$ and $b_1 = b$ and let p_1 be the midpoint of $[a, b]$; that is,

$$p_1 = \frac{a_1 + b_1}{2}$$

If $f(p_1) = 0$, then $p = p_1$, and we are done.

If $f(p_1) \neq 0$, then $f(p_1)$ has the same sign as either $f(a)$ or $f(b)$. If $f(p_1)$ and $f(a)$ have the same sign $P(p_1, b_1)$ set $a_2 = p_1$ and $b_2 = p_1$. Then re-apply the process to the interval $[a_2, b_2]$, ect

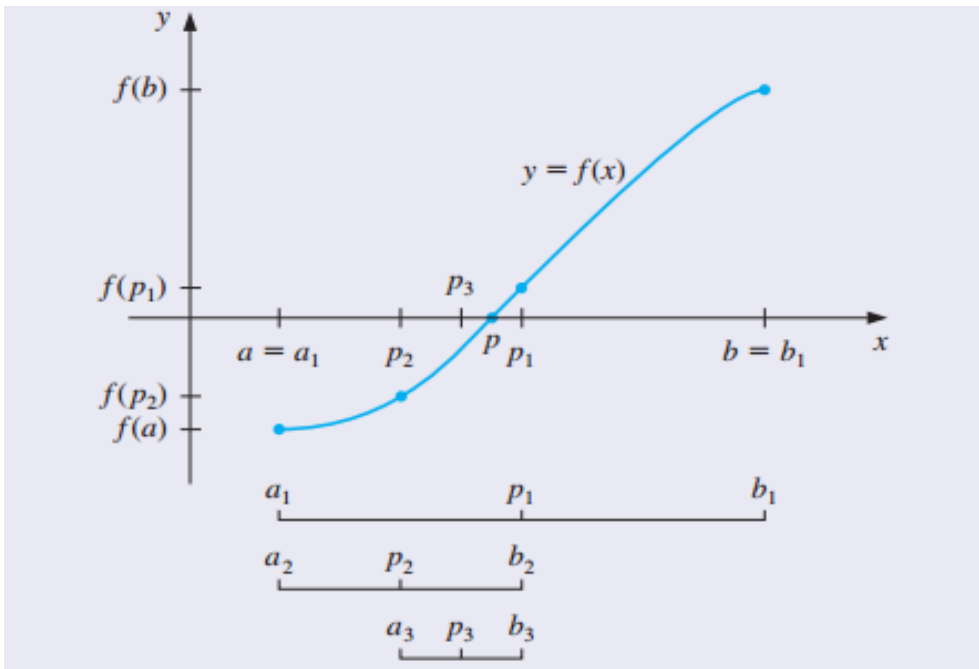


Fig No (1) Interval Halving to Bracket the Root

To find a solution to $f(x) = 0$ given the continuous function f on the interval $[a, b]$, where $f(a)$ and $f(b)$ have opposite signs:

INPUT endpoints a, b ; tolerance TOL; maximum number of iterations N_0

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$;

$FA = f(a)$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 set $p = a + (b - a)/2$; (Compute p .)

$FP = f(p)$.

Step 4 if $FP = 0$ or $(b - a)/2 < TOL$ then

OUTPUT (p); (Procedure completed successfully.)

STOP.

Step 5 Set $i = i + 1$.

Step 6 if $FA \cdot FP > 0$ then set $a = p$; (Compute a, b)

$FA = FP$

Else set $b = p$. (FA is unchanged)

Step 7 OUTPUT (Method failed after $N_0 = 'N_0$);

(the procedure was unsuccessful.) STOP.

Other stopping procedures can be applied in Step 4. For example, we can select a tolerance $\varepsilon > 0$ and generate p_1, \dots, p_N until one of the following conditions is met:

$$\begin{aligned} & |p_N - p_{N-1}| < \varepsilon \\ & \frac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon \quad p_N \neq 0 \quad \text{or} \end{aligned}$$

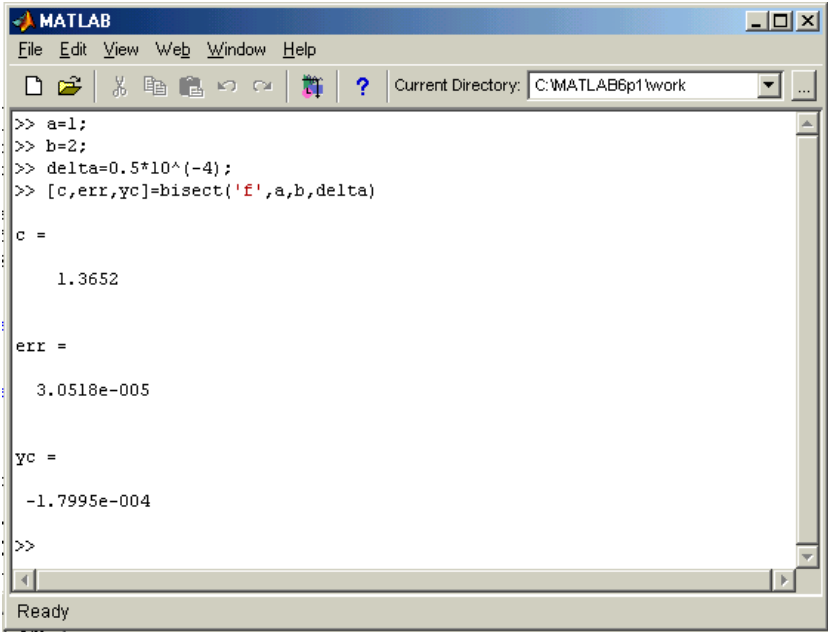
$|f(p_N)| < \varepsilon$ [8]

Example (6.1) Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$, and use the Bisection method to determine an approximation to the root that is accurate to at least within 10^{-4} using Matlab. [8]

Solution:

You must first know the function in m .file

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A screenshot of the MATLAB Command Window. The title bar reads 'MATLAB'. The menu bar includes 'File', 'Edit', 'View', 'Web', 'Window', and 'Help'. The toolbar contains icons for file operations and a 'Current Directory' dropdown set to 'C:\MATLAB6p1\work'. The command prompt shows the following code and output:

```
>> a=1;
>> b=2;
>> delta=0.5*10^(-4);
>> [c,err,yc]=bisect('f',a,b,delta)

c =

    1.3652

err =

    3.0518e-005

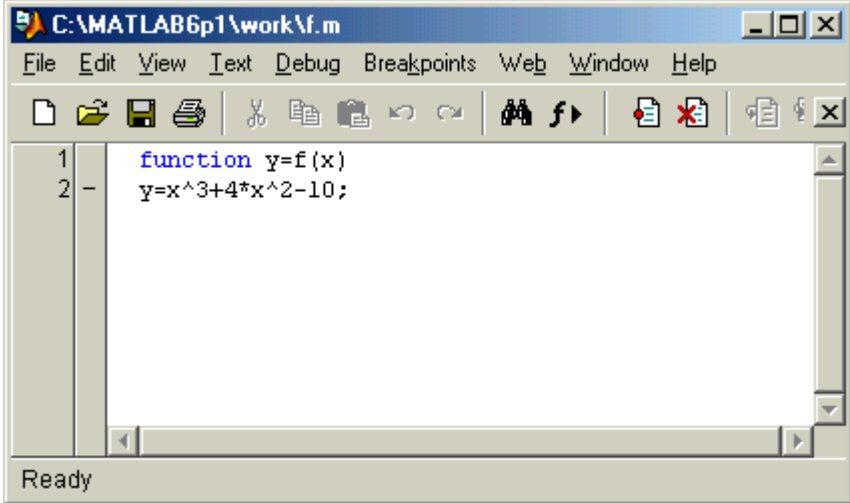
yc =

   -1.7995e-004

>>
```

The status bar at the bottom indicates 'Ready'.

Second: In the Command Window we write inputs



A screenshot of the MATLAB Editor window. The title bar reads 'C:\MATLAB6p1\work\mf.m'. The menu bar includes 'File', 'Edit', 'View', 'Text', 'Debug', 'Breakpoints', 'Web', 'Window', and 'Help'. The toolbar contains icons for file operations, editing, and execution. The editor shows the following code:

```
1 function y=f(x)
2 y=x^3+4*x^2-10;
```

The status bar at the bottom indicates 'Ready'.

Example (5.2): Use the bisection method with an Excel spreadsheet to approximate the value of $\sqrt{5}$ within 0.00001 accuracy using Matlab.[9]

Solution:

Finding the square root of 5 is equivalent to finding the roots of $x^2 - 5 = 0$. We expect the positive root to be in the $2 < x < 3$ interval so we assign $x_1 = 2$ $x_2 = 3$. The average of these values is $x_m = 2.5$ we can use it with any polynomial equation.

The screenshot shows the MATLAB Editor interface. The title bar reads "Editor - C:\Program Files\MATLAB\ccc\R2014a\bin\D.m". The menu bar includes EDITOR, PUBLISH, and VIEW. The toolbar contains icons for New, Open, Save, Find Files, Compare, Print, Insert, Comment, Indent, fx, and navigation tools like Go To and Find. The main workspace displays the following code:

```

1 | function y=f(x)
2 |     y=x^2-5;
3 | end
    
```

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Command Window

```
>> a=2;
>> b=3;
>> delta=10^(-5)

delta =

    1.0000e-05

>> [c,err,yc]=bisect('D',a,b,delta)

c =

    2.2361

err =

    7.6294e-06

yc =

   -1.3714e-05
```

Matlab Code :

Function [c,err,yc]=bisect(f,a,b,delta)

```
%Input - f is the function input as a string 'f'
%      -a and b are the left and right end points
%      -delta is the tolerance
%Output -c is the zero
%      -yc=f(c)
```

```

% -err is the error estimate for c
ya=feval(f,a);
yb=feval(f,b);
ifya*yb>0
end
max1=1+round((log(b-a)-log(delta))/log(2));
for k=1:max1
    c=(a+b)/2;
    yc=feval(f,c);
    ifyc==0
        a=c;
        b=c;
    elseifyb*yc>0
        b=c;
    yb=yc;
    else
        a=c;
    ya=yc;
    end
    if b-a<delta,break,end
end
c=(a+b)/2;
err=abs(b-a);
yc=feval(f,c);

```

Conclusion:

The goal of the bisection method which is studied in this paper is to locate the method stops if one of the midpoints happens to coincide with the root

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